Group Topologies and Semigroup Topologies on the Integers Determined by Convergent Sequences (General and Geometric Topology and its Applications)

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Abstract

We address the strongest group topologies and semigroup topologies on the integers with a certain sequence converging to 0 as investigated by Protasov and Zelenyuk. These are useful for constructing topological groups with peculiar duality properies and related to exponential Diophatine equations and additive bases.

1 Introduction

As in [10], for a group G and a sequence $\langle a_n : n \in \mathbf{N}\rangle$ of its elements, we denote by $(G|\langle a_n : n \in \mathbf{N}\rangle)$ the topological group G with the strongest group topology in which $\langle a_n : n \in \mathbf{N}\rangle$ converges to the neutral element; it is $G\{a_n\}$ in the notation of [14]. We admit non-Hausdorff topologies as well. Similarly for a monoid G, we denote by $(G|\langle a_n : n \in \mathbf{N}\rangle)_{\mathrm{s}}$ the topological semigroup with the strongest semigroup topology satisfying the convergence condition.

This article contains two themes concerning such topological (semi)groups as above; they are relatively independent each other. First we exhibit a pair of topological groups which witnesses that certain duality properties are not preserved under direct products (see [4] for details). In the second part, we observe additive properties of the integers through semigroup topologies.

In Section 2, we recall two duality properties we consider. Section 3 is for description of the counterexample. The (sketchy) proof of nonproductivity is completed in Section 4 invoking a theorem on exponential Diophantine equations. These constitute the first part. In Section 5, we characterize sequential convergence for such Abelian Hausdorff groups and pose a parallel problem for T_1 monoids. Some interconnection between T_1 semigroup topologies on the integers and and asymptotic bases are mentioned in Section 6.

Most of groups or semigroups we treat are commutative. For them, we adopt the additive notation and denote by 0 the neutral element, if any.

2 Duality Properties

All topological groups in Sections 2,3 and 4 should be Hausdorff and Abelian, and a character is a continuous homomorphism into the torus $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, unless otherwise stated. A subgroup H of a topological group G is *dually closed* if for each $g \in G \setminus H$, there exists a character χ of G that separates H and g, i.e., χ is identically zero on H and $\chi(g) \neq 0$. We say that H is *dually embedded* if each character of H extends to one of G.

Our concern is for the following two properties: "every closed subgroup is dually closed" and "every closed subgroup is dually embedded." We denote the former by X(1) and the latter by X(2) after [1].

There is misunderstanding in the literature ([9]) that each of the above is preserved under arbitrary direct products. We show that it is not the case.

3 Counterexample

Our counterexample consisits of $(\mathbf{Z}| \langle 2^n : n \in \mathbf{N}\rangle)$ and $(\mathbf{Z}| \langle 3^n : n \in \mathbf{N}\rangle)$. Their characters and closed subgroups are explicitly described in [8] and in [14].

We have rather straightforward observations:

1. Both groups have X(1) and X(2);
2. The diagonal $\Delta = \{\langle u, u\rangle : u \in \mathbf{Z}\} \subset (\mathbf{Z}| \langle 2^n : n \in \mathbf{N}\rangle) \times (\mathbf{Z}| \langle 3^n : n \in \mathbf{N}\rangle)$ and each element lying outside cannot be separated by the characters;
3. The group of characters of Δ extendable to the whole product is properly contained in the character group of Δ with the discrete topology.

In the next section, we see that Δ is discrete (and closed in the product). So it follows that the product has neither X(1) nor X(2).

4 Reduction to Number Theory

Through sequentiality argument, the following statement implies the discreteness of Δ:

> every sequence of integers converging to 0 both in $(\mathbf{Z}| \langle 2^n : n \in \mathbf{N}\rangle)$ and in $(\mathbf{Z}| \langle 3^n : n \in \mathbf{N}\rangle)$ is eventually equal to 0.

We shall establish this relying on two lemmata; one is number-theoretic and the other topological.

First recall a finiteness theorem for exponential Diophantine equations, a special case of [11, Ch. V, Theorem 2A]. Let S be a finite set of primes. A rational number is said to be an S-unit if it belongs to the multiplicative group generated by $S \cup \{-1\}$. The set of S-units is denoted by U_S.

Theorem 4.1 *Up to scalar multiplications, the equation $x_1 + \cdots + x_k = 0$ has only finitely many solutions $\langle x_1, ..., x_k\rangle$ in S-units whose non-trivial subsums do not vanish.* □

As a corollary, we also have finiteness for certain subsums.

Lemma 4.2 *Suppose that S and T are disjoint finite sets of primes. Let the tuple $\langle x_1, ..., x_k, y_1, ..., y_l\rangle$ run through the solutions of the equation $x_1 + \cdots + x_k = y_1 + \cdots + y_l$ with $x_i \in U_S \cup \{0\}$ and $y_j \in U_T \cup \{0\}$ for $1 \leq i \leq k, 1 \leq j \leq l$. Then the sum $x_1 + \cdots + x_k$ has only finitely many values.*

Next we need a result in [14] putting constraint on convergent sequences in topologial groups of the form $(G|\langle a_n : n \in \mathbf{N}\rangle)$.

Lemma 4.3 **([14, Lemma 2])** *If $g_m \to 0$ in $(G|\langle a_n : n \in \mathbf{N}\rangle)$, then there exists a positive integer k such that $g_m \in \{x_1 + \cdots + x_k : (\forall i)(x_i \in \{\pm a_n : n \in \mathbf{N}\} \cup \{0\})\}$ for sufficiently large m.* □

Now suppose that a sequence $\langle g_m : m \in \mathbf{N}\rangle$ of integers converges to 0 in $(\mathbf{Z}|\langle 2^n : n \in \mathbf{N}\rangle)$ and in $(\mathbf{Z}|\langle 3^n : n \in \mathbf{N}\rangle)$. By Lemma 4.3, there exists k such that g_m is a sum of less than k numbers in $\{\pm 2^n : n \in \mathbf{N}\}$ and in $\{\pm 3^n : n \in \mathbf{N}\}$, respectively, for sufficiently large m. Due to Lemma 4.2, there are only finitely many such sums. Therefore g_m is eventually equal to 0. Thus we are done.

5 Characterizing Convergent Sequences

¿From now on, we address commutative topological (semi)groups which need not be Hausdorff. Let $\langle a_n : n \in \mathbf{N}\rangle$ be a sequence in a group or in a monoid. We adopt some notations as in [14]:

$$A_m = \{a_n : n \geq m\},$$
$$A_m^\circ = A_m \cup \{0\},$$
$$A_m^* = \pm A_m^\circ.$$

Lemma 4.3 may be restated as follows.

Lemma 5.1 **([14, Lemma 2])** *Suppose that $(G|\langle a_n : n \in \mathbf{N}\rangle)$ is a Hausdorff Abelian group and that $g_m \to 0$ therein. Then there exists a positive integer k such that $g_m \in \underbrace{A_0^* + \cdots + A_0^*}_{k}$ for sufficiently large m.* □

Improving the above, we have a necessary and sufficient condition for a sequence in $(G|\langle a_n : n \in \mathbf{N}\rangle)$ to converge to 0.

Theorem 5.2 *In a Hausdorff Abelian topological group of the form $(G|\langle a_n : n \in \mathbf{N}\rangle)$, a sequence $\langle g_m : m \in \mathbf{N}\rangle$ converges to 0 if and only if there exists a natural number k such that for every $u \in \mathbf{N}$, all g_m except for finitely many m belong to $\underbrace{A_u^* + \cdots + A_u^*}_{k}$.* □

For T_1 monoids, Lemma 5.1 has a counterpart.

Proposition 5.3 *If $(G|\langle a_n : n \in \mathbf{N}\rangle)_{\mathrm{s}}$ is a T_1 commutative monoid with $g_m \to 0$. Then there exists a natural number k such that $g_m \in \underbrace{A_0^\circ + \cdots + A_0^\circ}_{k}$ for sufficiently large m.* □

For *Hausdorff* commutative *cancellative* monoids, Theorem 5.2 has a parallel. But we do not know whether these assumptions are necessary.

Conjecture 5.4 Suppose that $(G|\langle a_n : n \in \mathbf{N}\rangle)_{\mathrm{s}}$ is a T_1 commutative monoid and $\langle g_m : m \in \mathbf{N}\rangle$ is a sequence therein converging to 0. Then there exist a natural number k such that $\underbrace{A_u^\circ + \cdots + A_u^\circ}$ for every $u \in \mathbf{N}$ includes all g_m but for finitely many m.

6 Additive Bases

Suppose that B is a set of natural numbers. Recall that B is, by definition, an *asymptotic basis of order h* if $\underbrace{B + \cdots + B}_{h}$ contains all but finitely many natural numbers (cf. [7]).

For $m \in \mathbf{N}$, let $B^{\circ}_{\geq m}$ denote the set $\{n \in B : n \geq m\} \cup \{0\}$. We treat an infinite subset B of the natural numbers and its incresing enumeration $\langle a_n : n \in \mathbf{N} \rangle$ interchangeably. Note that $A^{\circ}_m = B^{\circ}_{\geq a_m}$.

We have interrelations between asymptotic bases and semigroup topologies.

Theorem 6.1 *Let B and $\langle a_n : n \in \mathbf{N} \rangle$ be as above. Then the implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *hold.*

(1) *There exists a fixed natural number h such that $B^{\circ}_{\geq m}$ for each $m \in \mathbf{N}$ is an asymptotic basis of order h.*

(2) $(\mathbf{Z} | \langle a_n : n \in \mathbf{N} \rangle)_s = (\mathbf{Z} | \langle n : n \in \mathbf{N} \rangle)_s$.

(3) *Each $B^{\circ}_{\geq m}$ is an asymptotic basis of finite order.*

□

Remark 6.2 The (apparent?) difference between (1) and (3) is vaguely indicated in [2, p. 52]. We do not know whether these are truly inequivalent.

If Conjecture 5.4 is true, then (1) and (2) are equivalent.

Example 6.3 Let B be the set of the primes or of the k-th powers of the natural numbers for some positive integer k. Then (1) in Theorem 6.1 holds. This is observed in several ways as follows.

For powers, a short interval solution for Waring's problem ([12, Theorem 1], cf. its improvements [13], [5]) yields that for each exponent $k \in \mathbf{N}$ there is a natural number h and a function $n \mapsto u(n)$ with $\lim_{n \to \infty} u(n) = \infty$ such that a sufficiently large natural number n has a representation $n = m_1^k + \cdots + m_h^k$ with $m_1, ..., m_h > u(n)$.

As to primes, due to a short inverval version of Vinogradov's three-prime theorem ([3, Theorem A]), any set of the form $B^{\circ}_{\geq m}$ is an asymptotic basis of order 4.

Probabilistic arguments as in [6, Theorem 1] also yield the result for powers.

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